On a Group of Transformations which Occurs in the Problem of Several Bodies.

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Given a system of $n+1$ bodies consisting of a fixed body $(0, 0, 0; \mu)$ and n others $(x_i, y_i, z_i; m_i)$, mutually attracting one another by central forces varying directly as the masses and as any arbitrary function of the distance; to determine the motion of the n bodies about the fixed center we arrive at a system of $6n$ differential equations of the first order in the canonical form:

$$\left.\begin{array}{lll} \dfrac{dx_i}{dt}=-\dfrac{\partial F}{\partial \xi_i}, & \dfrac{dy_i}{dt}=-\dfrac{\partial F}{\partial \eta_i}, & \dfrac{dz_i}{dt}=-\dfrac{\partial F}{\partial \zeta_i}, \\ \dfrac{d\xi_i}{dt}=\dfrac{\partial F}{\partial x_i}, & \dfrac{d\eta_i}{dt}=\dfrac{\partial F}{\partial y_i}, & \dfrac{d\zeta_i}{dt}=\dfrac{\partial F}{\partial z_i}, \end{array}\right\} \quad (i=1, 2, \ldots, n), \tag{1}$$

where ξ_i, η_i, ζ_i are proportional to the projections of the velocities of the bodies on the axes of coordinates, and the function F is of the form

$$F=U-\sum_{i=1}^{n}\frac{\xi_i^2+\eta_i^2+\zeta_i^2}{2m_i}, \tag{2}$$

the force-function being designated by U.

Let new variables

$$\left.\begin{array}{ll} q_{ij}=x_i x_j+y_i y_j+z_i z_j, & q_{ji}=q_{ij}, \\ r_{ij}=\xi_i \xi_j+\eta_i \eta_j+\zeta_i \zeta_j, & r_{ji}=r_{ij}, \\ s_{ij}=x_i \xi_j+y_i \eta_j+z_i \zeta_j, & s_{ji}\neq s_{ij}, \end{array}\right\} \quad (i, j=1, 2, 3, \ldots, n), \tag{3}$$

be introduced. These variables are of the same form as those employed by Bertrand* in the problem of three bodies. They are $n(2n+1)$ in number and are not all distinct. In fact we readily see from their form that the symmetrical determinant

$$\Delta=\begin{vmatrix} q_{ij} & s_{ij} \\ s_{ij} & r_{ij} \end{vmatrix}, \quad (i, j=1, 2, \ldots, n), \tag{4}$$

*Bertrand, Mémoire sur l'intégration des équations différentielles de la mécanique, *Journal de Liouville*, Ser. I, Vol. XVII (1852), pp. 393–436.

where q_{ij} represents the square array of n^2 elements obtained by giving to i, j the values $1, 2, \ldots, n$, and all its minors down to and including all the $\frac{1}{2}\binom{2n}{4}\left\{\binom{2n}{4}+1\right\}$ which are determinants of the fourth order vanish, and that no one of the $\frac{1}{2}\binom{2n}{3}\left\{\binom{2n}{3}+1\right\}$ which are of the third order vanishes. These $\frac{1}{2}\binom{2n}{4}\left\{\binom{2n}{4}+1\right\}$ conditions among $n(2n+1)$ quantities are far too numerous; they can be reduced to proper bounds by means of a theorem of Kronecker.* We find in fact that the vanishing of all the $\frac{1}{2}\binom{2n}{4}\left\{\binom{2n}{4}+1\right\}$ fourth order subdeterminants of the above symmetrical determinant is a consequence of the vanishing of $(n-1)(2n-3)$ properly chosen independent fourth order subdeterminants, and this choice can be made in $\frac{1}{2}\binom{2n}{3}\left\{\binom{2n}{3}+1\right\}$ ways. Then by the aid of these independent relations $(n-1)(2n-3)$ of the variables (3) can be eliminated if they be employed in problem (1); there would remain $6n-3$ independent variables, which would be sufficient since a loss of three from the original $6n$ independent variables can be accounted for by a change in orientation. On making $n=2$ in Δ we have Bour's determinant† the vanishing of which expresses the single relation among Bertrand's ten variables (3) in the problem of three bodies.

In the variables (3) the force-function U becomes

$$U=\sum_{i=1}^{n}\mu m_i f(\sqrt{q_{ii}})-\sum_{i=1}^{n}\sum_{j=1}^{n} m_i m_j f(\sqrt{q_{ii}+q_{jj}-2q_{ij}}); \tag{5}$$

accordingly the partial derivatives of F are of the form

$$\frac{\partial F}{\partial x_i}=\mu_i x_i+\sum_{j=1}^{n}\mu_{ij}x_j, \quad \frac{\partial F}{\partial \xi_i}=-\frac{\xi_i}{m_i}, \tag{6}$$

where the quantities

$$\left.\begin{aligned} \mu_i &= \mu m_i \frac{f'(\sqrt{q_{ii}})}{\sqrt{q_{ii}}}-\sum_{j=1}^{n}\mu_{ij}, \\ \mu_{ij} &= m_i m_j \frac{f'(\sqrt{q_{ii}+q_{jj}-2q_{ij}})}{\sqrt{q_{ii}+q_{jj}-2q_{ij}}}=\mu_{ji} \end{aligned}\right\} \tag{7}$$

are coefficients depending on the forces and expressed in terms of the q's alone.

Then in virtue of (1) the variables (3) satisfy the following system of ordinary differential equations:

$$\left.\begin{aligned} \frac{dq_{ij}}{dt} &= \frac{s_{ij}}{m_j}+\frac{s_{ji}}{m_i}, \\ \frac{dr_{ij}}{dt} &= \mu_i s_{ij}+\mu_j s_{ji}+\mu_{ij}(s_{ii}+s_{jj})+\sum_{k=1}^{n}\mu_{jk}s_{ki}+\sum_{l=1}^{n}\mu_{il}s_{lj}, \\ \frac{ds_{ij}}{dt} &= \mu_j q_{ij}+\mu_{ij}q_{ii}+\frac{r_{ij}}{m_i}+\sum_{k=1}^{n}\mu_{jk}q_{ik}; \end{aligned}\right\} (i, j=1, 2, \ldots, n), \tag{8}$$

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* Kronecker, Bemerkungen zur Determinanten-Theorie, *Crelle's Journal*, Vol. LXXII (1870), pp. 152–175.

† Bour, Mémoire sur le problème des trois corps, *Journal de l'École Polytechnique*, Vol. XXI (1856), pp. 35–58.

these equations are the generalizations of Bour's equations in the problem of three bodies.

If now, in this problem of the motion of n bodies about a fixed center under forces varying as an arbitrary function of the distance as formulated above, we seek those integrals which do not involve the law of force, we have to find those functions ϕ of all the q's, r's and s's not containing the μ's, whose total derivative with regard to the time

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left\{\frac{\partial\phi}{\partial q_{ij}}\frac{dq_{ij}}{dt}+\frac{\partial\phi}{\partial r_{ij}}\frac{dr_{ij}}{dt}+\frac{\partial\phi}{\partial s_{ij}}\frac{ds_{ij}}{dt}\right\} \tag{9}$$

vanishes independently of the μ's when the total derivatives are replaced by their values (8).

From the absolute term of the equation thus formed we have the equation

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left\{\left(\frac{s_{ij}}{m_j}+\frac{s_{ji}}{m_i}\right)\phi_{q_{ij}}+\frac{r_{ij}}{m_i}\phi_{s_{ij}}\right\}=0; \tag{10}$$

from the coefficients of the μ_i the following n equations:

$$b_i\equiv 2w_i\phi_{v_i}+u_i\phi_{w_i}+\sum_{j=1}^{n}(s_{ij}\phi_{r_{ij}}+q_{ij}\phi_{s_{ji}})=0,\quad (i=1,2,\ldots,n); \tag{11}$$

and finally from the terms in which the μ_{ij} appear the following $\frac{1}{2}n(n-1)$ equations:

$$\left.\begin{aligned}d_{ij}\equiv d_{ji}\equiv{}&2s_{ji}\phi_{v_i}+2s_{ij}\phi_{v_j}+q_{ij}(\phi_{w_i}+\phi_{w_j})+(w_i+w_j)\phi_{r_{ij}}+u_i\phi_{s_{ij}}+u_j\phi_{s_{ji}}\\&+\sum_{k=1}^{n}(s_{ik}\phi_{r_{jk}}+s_{jk}\phi_{r_{ik}}+q_{jk}\phi_{s_{ki}}+q_{ki}\phi_{s_{kj}})=0,\quad(i,j=1,2,\ldots,n),\end{aligned}\right\} \tag{12}$$

where for brevity we have put

$$q_{ii}=u_i,\qquad r_{ii}=v_i,\qquad s_{ii}=w_i. \tag{13}$$

Combining these $\frac{1}{2}n(n+1)+1$ equations (10), (11), (12) in all possible pairs, by Poisson's operation, we obtain the following complete system of $n(2n+1)$ linear partial differential equations of the first order:

$$\left.\begin{aligned}&a_i\equiv 2w_i\phi_{u_i}+v_i\phi_{w_i}+\sum_{j=1}^{n}(s_{ji}\phi_{q_{ij}}+r_{ij}\phi_{s_{ij}})=0;\quad b_i=0;\\&c_i\equiv 2u_i\phi_{u_i}-2v_i\phi_{v_i}+\sum_{j=1}^{n}(q_{ij}\phi_{q_{ij}}-r_{ij}\phi_{r_{ij}}+s_{ij}\phi_{s_{ij}}-s_{ji}\phi_{s_{ji}})=0;\quad d_{ij}=0;\\&e_{ij}\equiv 2q_{ij}\phi_{u_i}-2r_{ij}\phi_{v_j}+s_{ji}(\phi_{w_i}-\phi_{w_j})+u_j\phi_{q_{ij}}-v_i\phi_{r_{ij}}+(w_j-w_i)\phi_{s_{ij}}\\&\qquad+\sum_{k=1}^{n}(q_{jk}\phi_{q_{ik}}-r_{ki}\phi_{r_{jk}}+s_{jk}\phi_{s_{ik}}-s_{ki}\phi_{s_{kj}})=0;\\&f_{ij}\equiv 2s_{ij}\phi_{u_i}+2s_{ji}\phi_{u_j}+r_{ij}(\phi_{w_i}+\phi_{w_j})+(w_i+w_j)\phi_{q_{ij}}+v_j\phi_{s_{ij}}+v_i\phi_{s_{ji}}\\&\qquad+\sum_{k=1}^{n}(s_{kj}\phi_{q_{ik}}+s_{ki}\phi_{q_{jk}}+r_{jk}\phi_{s_{ik}}+r_{ki}\phi_{s_{jk}})=0;\\&\qquad d_{ji}=d_{ij},\qquad e_{ji}\mp e_{ij},\qquad f_{ji}=f_{ij},\qquad(i,j=1,2,\ldots,n).\end{aligned}\right\} \tag{14}$$

These equations are the generalizations of those given by Gravé* for the case $n = 2$.

The preceding remarks are taken from a previous paper,† in which was noted incidentally the fact that the $n(2n + 1)$ operators

$$a_i, \ b_i, \ c_i, \ d_{ij}, \ e_{ij}, \ f_{ij} \tag{15}$$

constitute a continuous group of transformations in Lie's sense. That these infinitesimal transformations generate a group was pointed out in an unpublished paper read by the writer before the American Mathematical Society, December 29, 1902.

It is the object of the present note to construct the invariants of this group by the methods of Sophus Lie; the variables will be regarded as independent, but for convenience the notation will not be changed.

The group property of the family (15) is exhibited by the following table of values of the symbol of Poisson:

$$\left.\begin{array}{llll}
(a_i, b_i) = -c_i; & (a_i, c_i) = 2a_i; & (a_i, d_{ij}) = -e_{ij}; & (a_i, e_{ji}) = f_{ij}; \\
(b_i, c_i) = -2b_i; & (b_i, e_{ij}) = -d_{ij}; & (b_i, f_{ij}) = e_{ji}; & \\
(c_i, d_{ij}) = d_{ij}; & \multicolumn{3}{l}{(c_i, e_{ij}) = -(c_j, e_{ij}) = -e_{ij}; \quad (c_i, e_{ji}) = -(c_j, e_{ji}) = e_{ji};} \\
(c_i, f_{ij}) = -f_{ij}; & & & \\
(d_{ij}, e_{ij}) = -2b_j; & (d_{ij}, e_{jk}) = -d_{ki}; & (d_{ij}, f_{ij}) = c_i + c_j; & \\
(d_{ij}, f_{jk}) = e_{kj}; & & & \\
(e_{ij}, e_{jk}) = -e_{ki}; & (e_{ij}, e_{ki}) = e_{kj}; & (e_{ij}, e_{ji}) = c_j - c_i; & \\
(e_{ij}, f_{ij}) = -2a_i; & (e_{ij}, f_{jk}) = -f_{ki}; & &
\end{array}\right\} \tag{16}$$

from which have been omitted both those which are identically zero and those which are the simple inverses of those given.

To find the invariants of the group we have only to integrate the complete system

$$a_i = b_i = c_i = d_{ij} = e_{ij} = f_{ij} = 0, \qquad (i, j = 1, 2, \ldots, n), \tag{17}$$

of linear partial differential equations of the first order.

This system (17) consists of $n(2n+1)$ equations in as many variables; hence that a solution exist it is necessary that the determinant of the coefficients of the partial derivatives should vanish. This determinant of the $n(2n + 1)$th order

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*Gravé, Sur le problème des trois corps, *Nouvelles Annales de Mathématiques*, Ser. III, Vol. XV (1896), pp. 537–547.

†Lovett, On a problem including that of several bodies and admitting of an additional integral, *Transactions of the American Mathematical Society*, Vol. VI (1905), pp. 491–495.

refuses to yield to the ordinary methods of evaluation. The question of the vanishing of the determinant, however, takes care of itself, for after a few reductions of the order of the system we shall find the condition of integrability satisfied. Moreover we may convince ourselves of the integrability of the system by remarking that the determinant Δ is a solution of the system (17), as may be verified immediately by the aid of the fundamental theorems in the expansion of determinants. Furthermore it is easy to convince one's self that there are not more than n solutions. In fact, writing down the determinant of the coefficients of the partial derivatives of the system (17), it appears that there is a determinant of order $2n^2$ in it whose principal diagonal is

$$w_1^2 w_2^2 \ldots . w_n^2 \prod_{ij} (w_i^2 - w_j^2)^2 \tag{18}$$

and unique. The existence of this unique term can not be used to prove the non-vanishing of a subdeterminant of a higher order, since its coefficient in the original determinant is a determinant all of whose elements are zero. We infer then that the system of the $n(2n+1)$th order has at most n solutions.

Let us now apply the method of Boole and Korkine to the reduction of the order of the system in hand.

The equation

$$a_1 \equiv 2w_1 \Phi_{u_i} + v_1 \Phi_{w_i} + \sum_1^n{}_j (s_{ji}\Phi_{q_{ij}} + r_{ij}\Phi_{s_{ij}}) = 0, \qquad (j \neq 1), \tag{19}$$

is equivalent to the simultaneous system of total differential equations

$$\frac{du_1}{2w_1} = \frac{dw_1}{v_1} = \frac{dq_{12}}{s_{21}} = \frac{dq_{13}}{s_{31}} = \ldots . = \frac{dq_{1n}}{s_{n1}} = \frac{ds_{12}}{r_{12}} = \frac{ds_{13}}{r_{13}} = \ldots . = \frac{ds_{1n}}{r_{1n}}, \tag{20}$$

of which we have the following $2n-1$ independent integrals:

$$\left.\begin{array}{c} u_1 v_1 - w_1^2 = \xi_1, \quad w_1 r_{1i} - v_1 s_{1i} = \xi_{1i}, \quad s_{1i} s_{i1} - q_{1i} r_{1i} = \eta_{1i}, \\ \xi_{1i} \neq \xi_{i1}, \quad \eta_{1i} = \eta_{i1}, \quad (i = 2, 3, \ldots, n). \end{array}\right\} \tag{21}$$

Introducing the quantities (21) as new variables into the remaining equations of the system (17) with a view to eliminating the $2n$ old variables

$$u_1, \; w_1, \; q_{12}, \; q_{13}, \ldots, \; q_{1n}, \; s_{12}, \; s_{13}, \ldots, \; s_{1n}, \tag{22}$$

there results a system of $(n+1)(2n-1)$ equations in which the variables (22) do not appear explicitly. Indicating by an upper index unity the result of the substitution (21) the first member of the last-mentioned system is the equation

$$a_2^1 = 0, \tag{23}$$

which is equivalent to the following simultaneous system:

$$\frac{du_2}{2w_2}=\frac{dw_2}{v_2}=\frac{dq_{23}}{s_{32}}=\frac{dq_{24}}{s_{42}}=\ldots\ldots=\frac{dq_{2n}}{s_{n2}}=\frac{ds_{21}}{r_{12}}=\frac{ds_{23}}{r_{23}}=\frac{ds_{24}}{r_{24}}=\ldots\ldots=\frac{ds_{2n}}{r_{2n}}, \tag{24}$$

of which we have the $2(n-1)$ independent algebraic integrals

$$\left.\begin{array}{l} u_2 v_2 - w_2^2 = \xi_2, \quad w_2 r_{i2} - v_2 s_{2i} = \xi_{2i}, \quad s_{2j} s_{j2} - q_{2j} r_{2j} = \eta_{2j}, \\ \xi_{i2} \neq \xi_{2i}, \quad \eta_{j2} = \eta_{2j}, \quad (i = 1, 3, 4, \ldots, n;\ j = 3, 4, \ldots, n). \end{array}\right\} \tag{25}$$

If the latter be employed as new variables to eliminate the $2n-1$ old ones,

$$u_2, \quad w_2, \quad q_{2i}, \quad s_{21}, \quad s_{2i}, \qquad (i = 3, 4, \ldots, n), \tag{26}$$

there results a system of $2n^2+n-2$ equations whose inital member is

$$a_3^{12} = 0. \tag{27}$$

Repeating this process $n-1$ times we arrive at the system whose first member is

$$a_n^{123\ldots n-1} = 0, \tag{28}$$

the corresponding total differential system possessing the following n independent algebraic integrals:

$$u_n v_n - w_n^2 = \xi_n, \quad w_n r_{ni} - v_n s_{ni} = \xi_{ni}, \quad (i = 1, 2, \ldots, n-1). \tag{29}$$

Up to this point all of the variables

$$u_i,\ w_i,\ q_{ij},\ s_{ij},\ s_{ji}, \quad (i, j = 1, 2, \ldots, n), \tag{30}$$

have been eliminated except

$$u_n,\ w_n,\ s_{n1},\ s_{n2}, \ldots, s_{n\,n-1}. \tag{31}$$

On introducing the variables (29) to eliminate the variables (31) the original system (17) is reduced after this the nth substitution to the following system of $2n^2$ equations:

$$c_i^{12\ldots n} \equiv 2v_i \phi_{v_i} + \sum_{j=1}^{n} (r_{ij}\phi_{r_{ij}} + \xi_{ij}\phi_{\xi_{ij}} + \xi_{ji}\phi_{\xi_{ji}}) = 0, \quad (i = 1, 2, \ldots, n;\ j \neq i); \tag{32}$$

$$\begin{aligned} f_{ij}^{12\ldots n} \equiv & -2\xi_{ij}\,\phi_{\xi_i} - 2\xi_{ji}\,\phi_{\xi_j} + (r_{ij}^2 - v_i v_j)(\phi_{\xi_{ij}} + \phi_{\xi_{ji}}) - (\xi_{ij} + \xi_{ji})\,\phi_{\eta_{ij}} \\ & + \sum_{1}^{n}{}^{k} \Big\{ (r_{ij} r_{ki} - v_i r_{jk})\,\phi_{\xi_{ik}} + (r_{ij} r_{jk} - v_j r_{ik})\,\phi_{\xi_{jk}} \\ & + \frac{1}{v_k}(r_{jk}\xi_{ki} - r_{ki}\xi_{kj})(\phi_{\eta_{jk}} - \phi_{\eta_{ki}}) \Big\} = 0, \\ & \qquad (i, j = 1, 2, \ldots, n;\ k \neq i,\ k \neq j); \end{aligned} \tag{33}$$

$$\begin{aligned} (v_i b_i^{12\ldots i} + w_i c_i^{12\ldots i})^{i+1,\, i+2, \ldots, n} \equiv & -\sum_{1}^{n}{}^{j} \Big\{ \xi_{ij}\,\phi_{r_{ij}} + r_{ij}\,\xi_i\,\phi_{\xi_{ij}} \\ & + \frac{1}{r_{ij}}(v_i v_j \eta_{ij} - \xi_{ij}\xi_{ji})\,\phi_{\xi_{ji}} \Big\} = 0, \qquad (i = 1, 2, \ldots, n); \end{aligned} \tag{34}$$

$$(w_i f_{ij}^{12\dots i} - v_i e_{ji}^{12\dots i})^{i+1,\,i+2,\dots,\,n} \equiv 2v_i\, r_{ij} \Phi_{v_i} + v_i v_j \Phi_{r_{ij}} + \sum_1^n{}^k v_i\, r_{jk}\, \Phi_{r_{ki}}$$
$$+ 2r_{ij}\, \xi_i\, \Phi_{\xi_i} + \frac{2}{r_{ij}} (v_i v_j \eta_{ij} - \xi_{ij}\xi_{ji})\, \Phi_{\xi_j} + r_{ij}\, \xi_{ij}\, (\Phi_{\xi_{ij}} + \Phi_{\xi_{ji}})$$
$$+ \sum_1^n{}^k \{ v_i\, \xi_{kj}\, \Phi_{\xi_{ki}} + (r_{jk}\, \xi_{ij} - v_j\, \xi_{ik})\, \Phi_{\xi_{jk}} + (2r_{ij}\, \xi_{ik} - r_{ki}\, \xi_{ij})\, \Phi_{\xi_{ik}} \}$$
$$+ \left[r_{ij}\, \xi_i + \frac{1}{r_{ij}} (v_i v_j \eta_{ij} - \xi_{ij}\xi_{ji}) \right] \Phi_{\eta_{ij}} + \sum_1^n{}^k \left[\frac{r_{jk}}{v_k r_{ki}} (\xi_{ik}\xi_{ki} - v_k v_i \eta_{ki}) \right.$$
$$\left. - \frac{\xi_{ik}\,\xi_{kj}}{v_k} \right] (\Phi_{\eta_{jk}} - \Phi_{\eta_{ki}}) = 0, \quad (i, j = 1, 2, \dots, n;\ k \neq i,\ k \neq j); \quad (35)$$

$$[w_j (w_i f_{ij}^{12\dots i} - v_i e_{ji}^{12\dots i})^{i+1,\,i+2,\dots,\,n-1} - v_j (v_i d_{ij}^{12\dots i} + w_i e_{ij}^{12\dots i})^{i+1,\,i+2,\dots,\,n-1}]^n$$
$$\equiv 2v_i\ \xi_{ji}\ \Phi_{v_i} + 2v_j\, \xi_{ij}\, \Phi_{v_j} + \sum_1^n{}^k (v_i\, \xi_{jk}\, \Phi_{r_{ki}} + v_j\, \xi_{ik}\, \Phi_{r_{jk}}) + 2\xi_i\, \xi_{ji}\, \Phi_{\xi_i} + 2\xi_j\, \xi_{ij}\, \Phi_{\xi_j}$$
$$+ [v_i v_j (\xi_i + \eta_{ij}) + \xi_{ij}\xi_{ji}]\, \Phi_{\xi_{ij}} + [v_i v_j (\xi_j + \eta_{ij}) + \xi_{ij}\xi_{ji}]\, \Phi_{\xi_{ji}}$$
$$+ \sum_1^n{}^k \left\{ \frac{v_i}{r_{jk}} (\xi_{jk}\xi_{kj} - v_j v_k \eta_{jk})\, \Phi_{\xi_{ki}} + \frac{v_j}{r_{ki}} (\xi_{ki}\xi_{ik} - v_k v_i \eta_{ki})\, \Phi_{\xi_{kj}} + [2\xi_{ik}\,\xi_{ji} \right.$$
$$\left. + \frac{r_{ki}}{r_{ij}} (v_i v_j \eta_{ij} - \xi_{ij}\xi_{ji})]\, \Phi_{\xi_{ik}} + [2\xi_{jk}\,\xi_{ij} + \frac{r_{jk}}{r_{ij}} (v_i v_j \eta_{ij} - \xi_{ij}\xi_{ji})\, \Phi_{\xi_{jk}} \right\}$$
$$+ (\xi_i\,\xi_{ji} + \xi_j\,\xi_{ij})\, \Phi_{\eta_{ij}} + \sum_1^n{}^k \left\{ \frac{\xi_{jk}}{v_k r_{ki}} (v_k v_i \eta_{ki} - \xi_{ki}\xi_{ik}) - \frac{\xi_{ki}}{v_k r_{jk}} (v_j v_k \eta_{jk} \right.$$
$$\left. - \xi_{jk}\xi_{kj}) \right\} (\Phi_{\eta_{ik}} - \Phi_{\eta_{kj}}) = 0, \qquad (i, j = 1, 2, \dots, n;\ k \neq i,\ k \neq j); \qquad (36)$$

equations whose construction has been facilitated by relations such as the following:

$$\left.\begin{aligned}
w_i\, s_{ji} - q_{ij}\, v_i &= \frac{s_{ji}\,\xi_{ij} + v_i\,\eta_{ij}}{r_{ij}}, \\
r_{ik}\, s_{kj} - r_{jk}\, s_{ki} &= \frac{r_{jk}\,\xi_{ki} - r_{ki}\,\xi_{kj}}{v_k}, \\
s_{ij}\, s_{ki} - q_{ki}\, r_{ij} &= \frac{r_{ij}}{v_i\, r_{ki}} (v_i\, \eta_{ki} + s_{ki}\,\xi_{ik}) - \frac{s_{ki}\,\xi_{ij}}{v_i}, \\
q_{ij}\, s_{ki} - q_{ki}\, s_{ji} &= \frac{s_{ji}\, s_{ki}}{v_i\, r_{ij}\, r_{ki}} (r_{ij}\,\xi_{ik} - r_{ki}\,\xi_{ij}) + \frac{s_{ji}\,\eta_{ki}}{r_{ki}} - \frac{s_{ki}\,\eta_{ij}}{r_{ij}}, \\
&\qquad (i, j, k = 1, 2, 3, \dots, n),
\end{aligned}\right\} \quad (37)$$

where

$$\xi_i = u_i v_i - w_i^2, \quad \xi_{ij} = w_i\, r_{ij} - v_i\, s_{ij} \neq \xi_{ji}, \quad \eta_{ij} = s_{ij}\, s_{ji} - q_{ij}\, r_{ij} = \eta_{ji},$$
$$(i, j = 1, 2, \dots, n). \qquad (38)$$

The equations (32) themselves constitute a complete system of n equations in $\frac{1}{2}n(3n-1)$ partial derivatives; the system then possesses $\frac{3}{2}n(n-1)$ independent solutions and these are readily found to be

$$\left.\begin{aligned}&\lambda_{ij}=\rho_{ij}\,\rho_{ji}=\frac{v_i}{r_{ij}}\,\frac{v_j}{r_{ji}}=\frac{v_i\,v_j}{r_{ij}^2}=\lambda_{ji},\\ &\beta_{ij}=\frac{\xi_{ij}}{r_{ij}},\quad \beta_{ji}=\frac{\xi_{ji}}{r_{ij}},\quad \beta_{ji}\neq\beta_{ij},\qquad (i,j=1,2,\ldots,n).\end{aligned}\right\}\tag{39}$$

Introducing these variables into the remaining equations (33), (34), (35), (36), eliminating the $\frac{1}{2}n(3n-1)$ old variables

$$v_1,\,v_2,\ldots,\,v_n,\ r_{ij},\ \xi_{ij},\ \xi_{ji},\qquad (i,j=1,2,\ldots,n),\tag{40}$$

we obtain the following system:

$$(34)^\lambda\equiv\sum_1^{n}{}^{j}\{2\lambda_{ij}\,\beta_{ij}\,\Phi_{\lambda_{ij}}+(\xi_i+\beta_{ij}^2)\,\Phi_{\beta_{ij}}+\lambda_{ij}\,\eta_{ij}\,\Phi_{\beta_{ji}}\}=0,\qquad (i=1,2,\ldots,n);\tag{41}$$

$$\begin{aligned}(33)^\lambda\equiv&-2\beta_{ij}\,\Phi_{\xi_i}-2\beta_{ji}\,\Phi_{\xi_j}-(\beta_{ij}+\beta_{ji})\,\Phi_{\eta_{ij}}+\sum_1^{n}{}^{k}\frac{\lambda_{ij}}{\rho_{ijk}}(\beta_{ki}-\beta_{kj})(\Phi_{\eta_{jk}}-\Phi_{\eta_{ki}})\\ &+(1-\lambda_{ij})\,(\Phi_{\beta_{ij}}+\Phi_{\beta_{ji}})+\sum_1^{n}{}^{k}\left\{\left(1-\frac{\rho_{ijk}}{\lambda_{jk}}\right)\Phi_{\beta_{ik}}+\left(1-\frac{\rho_{ijk}}{\lambda_{ki}}\right)\Phi_{\beta_{jk}}\right\}=0,\\ &(i,j,k=1,2,\ldots,n;\ k\neq i,\ k\neq j);\end{aligned}\tag{42}$$

$$\begin{aligned}(35)^\lambda\equiv\ &2\xi_i\,\Phi_{\xi_i}+2(\lambda_{ij}\,\eta_{ij}-\beta_{ij}\,\beta_{ji})\,\Phi_{\xi_j}+(\xi_i+\lambda_{ij}\,\eta_{ij}-\beta_{ij}\,\beta_{ji})\,\Phi_{\eta_{ij}}\\ &+\sum_1^{n}{}^{k}\left\{\frac{\lambda_{ij}}{\rho_{ijk}}(\beta_{ik}\,\beta_{ki}-\lambda_{ki}\,\eta_{ki}-\beta_{ik}\,\beta_{kj})\,(\Phi_{\eta_{jk}}-\Phi_{\eta_{ki}})\right\}+2\lambda_{ij}\,(1-\lambda_{ij})\,\Phi_{\lambda_{ij}}\\ &+2\sum_1^{n}{}^{k}\lambda_{ki}\left(1-\frac{\rho_{ijk}}{\lambda_{jk}}\right)\Phi_{\lambda_{ki}}+\beta_{ij}\,(1-\lambda_{ij})\,\Phi_{\beta_{ij}}+(\beta_{ij}-\lambda_{ij}\,\beta_{ji})\,\Phi_{\beta_{ji}}\\ &+\sum_1^{n}{}^{k}\left\{\frac{\rho_{ijk}}{\lambda_{jk}}[(\beta_{kj}-\beta_{ki})\,\Phi_{\beta_{ki}}-\beta_{ik}\,\Phi_{\beta_{ik}}]+(2\beta_{ik}-\beta_{ij})\,\Phi_{\beta_{ik}}\right.\\ &\left.+\left(\beta_{ij}-\frac{\rho_{ijk}}{\lambda_{ki}}\beta_{ik}\right)\Phi_{\beta_{jk}}\right\}=0,\qquad (i,j,k=1,2,\ldots,n;\ k\neq i,\ k\neq j);\end{aligned}\tag{43}$$

$$
\begin{aligned}
(36)^\lambda \equiv\ & 2\xi_i\,\beta_{ji}\,\phi_{\xi_i} + 2\xi_j\,\beta_{ij}\,\phi_{\xi_j} + (\xi_i\,\beta_{ji} + \xi_j\,\beta_{ij})\,\phi_{\eta_{ij}} \\
& + \sum_1^n{}^k \frac{\lambda_{ij}}{\rho_{ijk}} \left[\beta_{jk}\,(\lambda_{ki}\,\eta_{ki} - \beta_{ki}\,\beta_{ik}) - \beta_{ik}\,(\lambda_{jk}\,\eta_{jk} - \beta_{jk}\,\beta_{kj})\right]\,(\phi_{\eta_{ki}} - \phi_{\eta_{jk}}) \\
& + 2\lambda_{ij}\,(\beta_{ij} + \beta_{ji})\,\phi_{\lambda_{ij}} + 2\sum_1^n{}^k \left[\lambda_{jk}\left(\beta_{ij} - \frac{\rho_{ijk}}{\lambda_{ki}}\beta_{ik}\right)\phi_{\lambda_{jk}} + \lambda_{ki}\left(\beta_{ji} - \frac{\rho_{ijk}}{\lambda_{jk}}\beta_{jk}\right)\phi_{\lambda_{ki}}\right] \\
& + [\lambda_{ij}\,(\xi_i + \eta_{ij}) + \beta_{ij}\,\beta_{ji}]\,\phi_{\beta_{ij}} + [\lambda_{ij}\,(\xi_j + \eta_{ij}) + \beta_{ij}\,\beta_{ji}]\,\phi_{\beta_{ji}} \\
& + \sum_1^n{}^k \Big\{\left(2\beta_{ij}\,\beta_{jk} - \frac{\rho_{ijk}}{\lambda_{ki}}\beta_{ik}\,\beta_{jk} + \lambda_{ij}\,\eta_{ij} - \beta_{ij}\,\beta_{ji}\right)\phi_{\beta_{jk}} \\
& + \frac{\rho_{ijk}}{\lambda_{kj}}\left(\beta_{jk}\,\beta_{kj} - \lambda_{jk}\,\eta_{jk} - \beta_{ki}\,\beta_{kj}\right)\phi_{\beta_{ki}} + \Big(2\beta_{ji}\,\beta_{ik} - \frac{\rho_{ijk}}{\lambda_{jk}}\beta_{ik}\,\beta_{jk} \\
& + \lambda_{ij}\,\eta_{ij} - \beta_{ij}\,\beta_{ji}\Big)\phi_{\beta_{ik}} + \frac{\rho_{ijk}}{\lambda_{ki}}\left(\beta_{ki}\,\beta_{ik} - \lambda_{ki}\,\eta_{ki} - \beta_{jk}\,\beta_{ik}\right)\phi_{\beta_{kj}}\Big\} = 0, \\
& \qquad (i,\,j,\,k = 1,\,2,\ldots,\,n;\; k \neq i,\, k \neq j); \qquad (44)
\end{aligned}
$$

where the upper index λ indicates the result of replacing (40) by (39), and where we have put

$$
\rho_{ijk} = \rho_{jki} = \rho_{kij} = \frac{v_i\,v_j\,v_k}{r_{ij}\,r_{jk}\,r_{ki}} = \sqrt{\rho_{ij}\,\rho_{jk}\,\rho_{ki}\,\rho_{ik}\,\rho_{kj}\,\rho_{ji}} = \sqrt{\lambda_{ij}\,\lambda_{jk}\,\lambda_{ki}}, \qquad (i,\,j,\,k = 1,\,2,\ldots,\,n). \qquad (45)
$$

The complete system (41), (42), (43), (44) consists of $n\,(2n-1)$ equations in $n\,(2n-1)$ variables

$$
\left.\begin{array}{lll}
\xi_1,\,\xi_2,\ldots,\,\xi_n; & \eta_{12},\ldots,\,\eta_{jk},\ldots,\,\eta_{n-1\,n}; & \\
\lambda_{12},\ldots,\,\lambda_{n-1\,n}; & \beta_{12},\ldots,\,\beta_{n-1\,n}; & \beta_{21},\ldots,\,\beta_{n-1\,n};
\end{array}\right\} (46)
$$

in order that the system have a solution it is necessary and sufficient that the determinant of the coefficients of the partial derivatives should vanish.

Let the equations be so written that the partial derivatives follow the order (46), and let the coefficients be

$$
\left.\begin{array}{lll}
\mathrm{X}_1^{(i)},\,\mathrm{X}_2^{(i)},\ldots,\,\mathrm{X}_n^{(i)}; & \mathrm{H}_{12}^{(i)},\ldots,\,\mathrm{H}_{n-1\,n}^{(i)}; & \\
\Lambda_{12}^{(i)},\ldots,\,\Lambda_{n-1\,n}^{(i)}; & \mathrm{B}_{12}^{(i)},\ldots,\,\mathrm{B}_{n-1\,n}^{(i)}; & \mathrm{B}_{21}^{(i)},\ldots,\,\mathrm{B}_{n\,n-1}^{(i)},
\end{array}\right\} (47)
$$
$$[i = 1,\,2,\ldots,\,n\,(2n-1)].$$

Then it appears at once from the equations that

$$
\sum_1^n{}^j \mathrm{X}_j^{(i)} - 2\sum^j\sum_1^n{}^k \mathrm{H}_{jk}^{(i)} = 0, \qquad [i = 1,\,2,\ldots,\,n\,(2n-1)], \qquad (48)
$$

for all values of i; hence the determinant vanishes.

The form of the equations (41) suggests the possibility of solutions in which the variables

$$\lambda_{ij},\ \beta_{ij},\ \beta_{ji}, \qquad (i, j = 1, 2, \ldots, n), \tag{49}$$

do not occur. To this end it is necessary and sufficient that all determinants of order $\frac{1}{2}n(n+1)$ in the matrix

$$\| X_1^{(i)}, X_2^{(i)}, \ldots, X_n^{(i)}, H_{12}^{(i)}, \ldots, H_{jk}^{(i)}, \ldots, H_{n-1\,n}^{(i)} \|, \ [i = 1, 2, \ldots, n(2n-1)], \tag{50}$$

should vanish; but all these determinants do vanish in consequence of (48). If the system corresponding to any of these determinants be taken, the solution is immediately found in the form

$$\sum_1^n{}^i\, \xi_i - 2\sum_1^n{}^i\sum{}^j\, \eta_{ij}; \tag{51}$$

this integral involves all of the original variables

$$u_i,\ v_i,\ w_i,\ q_{ij},\ r_{ij},\ s_{ij},\ s_{ji}, \quad (i, j = 1, 2, \ldots, n); \tag{52}$$

and hence we have an additional reason for the vanishing of the determinant of the complete system (17).

The further reduction of the complete system of equations which we have been studying is attended by serious complications. It is possible however to examine the most symmetrical case, namely that for which n is equal to three, more closely with comparative ease, and to show by a method which extends itself to the case of n arbitrary, that no other solutions exist than those already found.

When n equals three the reduced system of the $n(2n-1)$th order consists of fifteen equations which can be arranged in five sets of three equations each, the indices being permuted cyclically in each set; the equations for this case are as follows:

$$(v_1 b_1^1 + w_1 c_1^1)^{234} \equiv A_{123} = 0,\ (v_2 b_2^{12} + w_2 c_2^{12})^{34} \equiv A_{231} = 0,\ (v_3 b_3^{123} + w_3 c_3^{123})^4 \equiv A_{312} = 0; \tag{53}$$

$$f_{12}^{1234} \equiv B_{123} = 0, \qquad f_{23}^{1234} \equiv B_{231} = 0, \qquad f_{31}^{1234} \equiv B_{312} = 0; \tag{54}$$

$$(w_1 f_{12}^1 - v_1 e_{21}^1)^{234} \equiv C_{123} = 0,\ (w_2 f_{23}^{12} - v_2 e_{32}^{12})^{34} \equiv C_{231} = 0,\ (w_3 f_{31}^{123} - v_3 e_{13}^{123})^4 \equiv C_{312} = 0; \tag{55}$$

$$(w_1 f_{31}^1 - v_1 e_{31}^1)^{234} \equiv D_{123} = 0,\ (w_2 f_{12}^{12} - v_2 e_{12}^{12})^{34} \equiv D_{231} = 0,\ (w_3 f_{23}^{123} - v_3 e_{23}^{123})^4 \equiv D_{312} = 0; \tag{56}$$

$$\left.\begin{aligned} \{w_2 (w_1 f_{12}^1 - v_1 e_{21}^1)^2 - v_2 (v_1 c_1^1 + w_1 e_{12}^1)^2\}^{34} &\equiv E_{123} = 0, \\ \{w_3 (w_2 f_{23}^{12} - v_2 e_{32}^{12})^3 - v_3 (v_2 c_2^{12} + w_2 e_{23}^{12})^3\}^4 &\equiv E_{231} = 0, \\ \{w_3 (w_1 f_{31}^1 - v_1 e_{31}^1)^{23} - v_3 (v_1 c_3^1 + w_1 e_{13}^1)^{23}\}^4 &\equiv E_{312} = 0; \end{aligned}\right\} \tag{57}$$

where

$$\left.\begin{aligned}
A_{ijk} \equiv\ & 2\beta_{ij}\lambda_{ij}\Phi_{\lambda_{ij}} + 2\beta_{ik}\lambda_{ki}\Phi_{\lambda_{ki}} + \lambda_{ki}\eta_{ki}\Phi_{\beta_{ki}} + (\xi_i + \beta_{ij}^2)\Phi_{\beta_{ij}} + \lambda_{ij}\eta_{ij}\Phi_{\beta_{ji}} \\
& + (\xi_i + \beta_{ik}^2)\Phi_{\beta_{ik}}; \\
B_{ijk} \equiv\ & 2\beta_{ij}\Phi_{\xi_i} + 2\beta_{ji}\Phi_{\xi_j} + (\beta_{ij} + \beta_{ji})\Phi_{\eta_{ij}} + \frac{\lambda_{ij}}{\rho}(\beta_{kj} - \beta_{ki})(\Phi_{\eta_{jk}} - \Phi_{\eta_{ki}}) \\
& + (\lambda_{ij} - 1)(\Phi_{\beta_{ij}} + \Phi_{\beta_{ji}}) + \left(\frac{\rho}{\lambda_{ki}} - 1\right)\Phi_{\beta_{jk}} + \left(\frac{\rho}{\lambda_{jk}} - 1\right)\Phi_{\beta_{ik}}; \\
C_{ijk} \equiv\ & 2(\lambda_{ij}\eta_{ij} - \beta_{ij}\beta_{ji})\Phi_{\xi_i} + 2\xi_j\Phi_{\xi_j} + (\xi_j + \lambda_{ij}\eta_{ij} - \beta_{ij}\beta_{ji})\Phi_{\eta_{ij}} \\
& + \frac{\lambda_{ij}}{\rho}(\lambda_{jk}\eta_{jk} - \beta_{jk}\beta_{kj} + \beta_{jk}\beta_{ki})(\Phi_{\eta_{jk}} - \Phi_{\eta_{ki}}) + 2\lambda_{ij}(1 - \lambda_{ij})\Phi_{\lambda_{ij}} \\
& + 2\lambda_{jk}\left(1 - \frac{\rho}{\lambda_{ki}}\right)\Phi_{\lambda_{jk}} + (\beta_{ji} - \lambda_{ij}\beta_{ij})\Phi_{\beta_{ij}} + \beta_{ji}(1 - \lambda_{ij})\Phi_{\beta_{ji}} \\
& + \left(2\beta_{jk} - \beta_{ji} - \frac{\rho}{\lambda_{ki}}\beta_{jk}\right)\Phi_{\beta_{jk}} + \frac{\rho}{\lambda_{ki}}(\beta_{ki} - \beta_{kj})\Phi_{\beta_{kj}} + \left(\beta_{ji} - \frac{\rho}{\lambda_{jk}}\beta_{jk}\right)\Phi_{\beta_{ik}}; \\
D_{ijk} \equiv\ & C_{jik}; \\
E_{ijk} \equiv\ & 2\beta_{ji}\xi_i\Phi_{\xi_i} + 2\beta_{ij}\xi_j\Phi_{\xi_j} + (\beta_{ji}\xi_i + \beta_{ij}\xi_j)\Phi_{\eta_{ij}} \\
& + \frac{\lambda_{ij}}{\rho}\{\beta_{ik}(\lambda_{jk}\eta_{jk} - \beta_{jk}\beta_{kj}) - \beta_{jk}(\lambda_{ki}\eta_{ki} - \beta_{ki}\beta_{ik})\}(\Phi_{\eta_{jk}} - \Phi_{\eta_{ki}}) \\
& + 2\lambda_{ij}(\beta_{ij} + \beta_{ji})\Phi_{\lambda_{ij}} + 2\lambda_{jk}\left(\beta_{ij} - \frac{\rho}{\lambda_{ki}}\beta_{ik}\right)\Phi_{\lambda_{jk}} - 2\lambda_{ki}\left(\beta_{ji} - \frac{\rho}{\lambda_{jk}}\beta_{jk}\right)\Phi_{\lambda_{ki}} \\
& + [\lambda_{ij}(\xi_i + \eta_{ij}) + \beta_{ij}\beta_{ji}]\Phi_{\beta_{ij}} + \left(\lambda_{ij}\eta_{ij} - \beta_{ij}\beta_{ji} + 2\beta_{ij}\beta_{jk} - \frac{\rho}{\lambda_{ki}}\beta_{ik}\beta_{jk}\right)\Phi_{\beta_{jk}} \\
& + \frac{\rho}{\lambda_{jk}}(\beta_{kj}\beta_{jk} - \lambda_{jk}\eta_{jk} - \beta_{jk}\beta_{ki})\Phi_{\beta_{ki}} + (\lambda_{ij}\eta_{ij} - \beta_{ij}\beta_{ji} + 2\beta_{ji}\beta_{ik} \\
& - \frac{\rho}{\lambda_{jk}}\beta_{ik}\beta_{jk})\Phi_{\beta_{ik}} + \frac{\rho}{\lambda_{ki}}(\beta_{ki}\beta_{ik} - \lambda_{ki}\eta_{ki} - \beta_{ik}\beta_{kj})\Phi_{\beta_{kj}} \\
& + [\lambda_{ij}(\xi_j + \eta_{ij}) + \beta_{ij}\beta_{ji}]\Phi_{\beta_{ji}};
\end{aligned}\right\} \quad (58)$$

where

$$\rho = \sqrt{\lambda_{ij}\lambda_{jk}\lambda_{ki}}. \tag{59}$$

That the above fifteen equations in fifteen variables possess at least one solution appears from the fact that we have

$$P_{ijk} + P_{jki} + P_{kij} = 0, \tag{60}$$

where

$$P_{ijk} \equiv A_{ijk} + \frac{1}{\lambda_{ij}}\{(\beta_{ij}\beta_{ji} - \lambda_{ij}\eta_{ij})B_{ijk} + \beta_{ij}C_{ijk} + \beta_{ji}D_{ijk} - E_{ijk}\}. \tag{61}$$

That they form a complete system is verified by reference to the table below, from which have been omitted all vanishing commutators and all cyclical changes of those given:

$$
\left.
\begin{aligned}
&(A_{ijk},\, B_{ijk}) \equiv \beta_{ij}\, B_{ijk} + D_{ijk};\quad (A_{ijk},\, B_{kij}) \equiv \beta_{ik}\, B_{kij} + C_{kij};\\
&(A_{ijk},\, C_{ijk}) \equiv \beta_{ij}\, C_{ijk} - E_{ijk};\quad (A_{ijk},\, C_{kij}) \equiv \beta_{ik}\, C_{kij} - \xi_i\, B_{kij};\\
&(A_{ijk},\, D_{ijk}) \equiv \beta_{ij}\, C_{ijk} - E_{ijk};\quad (A_{ijk},\, D_{kij}) \equiv \beta_{ik}\, D_{kij} - E_{kij};\\
&(A_{ijk},\, E_{ijk}) \equiv \xi_i\, C_{ijk} + \beta_{ij}\, E_{ijk} - 4\,\beta_{ji}\, A_{ijk};\quad (A_{ijk},\, E_{kij}) \equiv \xi_i\, D_{kij} + \beta_{ik}\, E_{kij}\\
&\qquad\qquad - 4\,\beta_{ki}\, A_{ijk};\\
&(B_{ijk},\, C_{ijk}) \equiv (1+\lambda_{ij})\, B_{ijk} \equiv (B_{ijk},\, D_{ijk});\; (B_{ijk},\, C_{kij}) \equiv B_{kij} + \frac{\rho}{\lambda_{jk}}\,(B_{ijk} - B_{jki});\\
&(B_{ijk},\, D_{jki}) \equiv B_{jki} + \frac{\rho}{\lambda_{ki}}\,(B_{ijk} - B_{kij});\quad (B_{ijk},\, E_{ijk}) \equiv - C_{ijk} - D_{ijk};\\
&(B_{ijk},\, E_{jki}) \equiv \frac{\rho}{\lambda_{ki}}\,\beta_{ki}\, B_{ijk} - C_{jki} + \frac{\rho}{\lambda_{ki}}\, D_{kij};\quad (B_{ijk},\, E_{kij}) \equiv \frac{\rho}{\lambda_{jk}}\,\beta_{kj}\, B_{ijk} + C_{jki}\\
&\qquad\qquad - \frac{\rho}{\lambda_{jk}}\, D_{kij};\\
&(C_{ijk},\, C_{jki}) \equiv \beta_{kj}\, B_{ijk} + \frac{\rho}{\lambda_{ki}}\,(D_{kij} - C_{jki});\quad (C_{ijk},\, D_{ijk}) \equiv (\beta_{ij} - \beta_{ji})\, B_{ijk}\\
&\qquad\qquad + \lambda_{ij}\,(C_{ijk} - D_{ijk});\\
&(C_{ijk},\, D_{jki}) \equiv \beta_{ji}\, B_{jki} - \beta_{jk}\, B_{ijk} + \left(\frac{\rho}{\lambda_{ki}} - 2\right)(C_{ijk} - D_{jki});\\
&(C_{ijk},\, E_{ijk}) \equiv \lambda_{ij}\,(A_{ijk} + 3\,A_{jki}) + (\beta_{ij} - \beta_{ji})\, C_{ijk} + (1 - \lambda_{ij})\, E_{ijk};\\
&(C_{ijk},\, E_{jki}) \equiv (\lambda_{jk}\,\eta_{jk} - \beta_{jk}\,\beta_{kj})\, B_{ijk} + \left(\frac{\rho}{\lambda_{ki}}\beta_{ki} - 2\,\beta_{kj}\right) C_{ijk} - \beta_{ji}\, C_{jki}\\
&\qquad\qquad + \left(2 - \frac{\rho}{\lambda_{ki}}\right) E_{jki};\\
&(C_{ijk},\, E_{kij}) \equiv \frac{\rho}{\lambda_{jk}}\beta_{kj}\, C_{ijk} + \beta_{ji}\, D_{kij} - \frac{\rho}{\lambda_{jk}}\, E_{jki};\\
&(D_{ijk},\, D_{jki}) \equiv -\beta_{ij}\, B_{jki} - \frac{\rho}{\lambda_{ki}}\,(C_{kij} - D_{ijk});\\
&(D_{ijk},\, D_{kij}) \equiv \beta_{ki}\, B_{ijk} + \frac{\rho}{\lambda_{jk}}\,(C_{jki} - D_{kij});\\
&(D_{ijk},\, E_{ijk}) \equiv \lambda_{ij}\,(3\,A_{ijk} + A_{jki}) + (\beta_{ij} - \beta_{ji})\, D_{ijk} + (1 - \lambda_{ij})\, E_{ijk};\\
&(D_{ijk},\, E_{jki}) \equiv \beta_{ij}\, C_{jki} + \frac{\rho}{\lambda_{ki}}\beta_{ki}\, D_{ijk} - \frac{\rho}{\lambda_{ki}}\, E_{kij};\\
&(D_{ijk},\, E_{kij}) \equiv (\lambda_{ki}\,\eta_{ki} - \beta_{ki}\,\beta_{ik})\, B_{ijk} + \left(\frac{\rho}{\lambda_{jk}}\,\beta_{kj} - 2\,\beta_{ki}\right) D_{ijk} - \beta_{ij}\, D_{kij}\\
&\qquad\qquad + \left(2 - \frac{\rho}{\lambda_{jk}}\right) E_{kij};\\
&(E_{ijk},\, E_{jki}) \equiv (\lambda_{ij}\,\eta_{ij} - \beta_{ij}\,\beta_{ji})\, C_{jki} - (\lambda_{jk}\,\eta_{jk} - \beta_{jk}\,\beta_{kj})\, D_{ijk}\\
&\qquad\qquad + \left(\frac{\rho}{\lambda_{ki}}\,\beta_{ki} - 2\,\beta_{kj}\right) E_{ijk} + \left(2\,\beta_{ij} - \frac{\rho}{\lambda_{ki}}\,\beta_{ik}\right) E_{jki};\\
&(E_{ijk},\, E_{kij}) \equiv (\beta_{ki}\,\beta_{ik} - \lambda_{ki}\,\eta_{ki})\, C_{ijk} + (\lambda_{ij}\,\eta_{ij} - \beta_{ij}\,\beta_{ji})\, D_{kij}\\
&\qquad\qquad + \left(\frac{\rho}{\lambda_{jk}}\,\beta_{kj} - 2\,\beta_{ki}\right) E_{ijk} + \left(2\,\beta_{ji} - \frac{\rho}{\lambda_{jk}}\,\beta_{jk}\right) E_{kij}.
\end{aligned}
\right\}\quad (62)
$$

The system is known to possess two solutions, namely (4) and (51) from the preceding discussion. The latter of these is

$$\xi_i + \xi_j + \xi_k - 2(\eta_{ij} + \eta_{jk} + \eta_{ki}); \tag{63}$$

the former may be written

$$\begin{vmatrix} u_i & w_i & q_{ji} & s_{ij} & q_{ki} & s_{ik} \\ w_i & v_i & s_{ji} & r_{ij} & s_{ki} & r_{ik} \\ q_{ij} & s_{ji} & u_j & w_j & q_{kj} & s_{jk} \\ s_{ij} & r_{ji} & w_j & v_j & s_{kj} & r_{jk} \\ q_{ik} & s_{ki} & q_{jk} & s_{kj} & u_k & w_k \\ s_{ik} & r_{ki} & s_{jk} & r_{kj} & w_k & v_k \end{vmatrix} \tag{64}$$

Designate by 123456 the columns of the matrix formed by the first two rows of this determinant (64); those of the matrix formed by the third and fourth rows by $1'2'3'4'5'6'$; and finally the columns of the matrix formed by the last two rows by $1_1 2_1 3_1 4_1 5_1 6_1$; call these matrices A, B, C respectively.

The expressions of the minors of any one of them, say A, in the new variables, can be determined by means of the relations (37) and (38); the expressions of the remaining ones may then be written down by cyclical permutation guided by the following substitution scheme:

$$A(123456) \quad B(3'4'5'6'1'2') \quad C(5_1 6_1 1_1 2_1 3_1 4_1); \tag{65}$$

and the expansion of the determinant by the method of Laplace be obtained by substitution in the symbolical scheme below:

$$\left.\begin{aligned} &12\{3456 + 4536 + 5634 - 3546 - 4635 + 3645\} \\ &+ 23\{1456 + 4516 + 5614 - 1546 - 4615 + 1645\} \\ &+ 34\{1256 + 2516 + 5612 - 1526 - 2615 + 1625\} \\ &+ 45\{1236 + 2316 + 3612 - 1326 - 2613 + 1623\} \\ &+ 56\{1234 + 2314 + 3412 - 1324 - 2413 + 1423\} \\ &- 13\{2456 + 4526 + 5624 - 2546 - 4625 + 2645\} \\ &- 24\{1356 + 3516 + 5613 - 1536 - 3615 + 1635\} \\ &- 35\{1246 + 2416 + 4612 - 1426 - 2614 + 1624\} \\ &- 46\{1235 + 2315 + 3512 - 1325 - 2513 + 1523\} \\ &+ 14\{2356 + 3526 + 5623 - 2536 - 3625 + 2635\} \\ &+ 25\{1346 + 3416 + 4613 - 1436 - 3614 + 1634\} \\ &+ 36\{1245 + 2415 + 4512 - 1425 - 2514 + 1524\} \\ &- 15\{2346 + 3426 + 4623 - 2436 - 3624 + 2634\} \\ &- 26\{1345 + 3415 + 4513 - 1435 - 3514 + 1534\} \\ &+ 16\{2345 + 3425 + 4523 - 2435 - 3524 + 2534\}, \end{aligned}\right\} \tag{66}$$

in which the numbers without the parentheses belong to A, and of those within the parentheses the first pair of each set belongs to B and the second pair to C; the resulting form is long and complicated, the elegance of the form (64) disappearing in the transformation, and as it is unnecessary to our purposes it need not be reproduced here.

It is now proposed to show by the aid of the determinant of the partial derivatives of the system (58) that the system composed of (53), (54), (55), (56), (57) does not possess more than two solutions.

Call the determinant D and write it down so that its fifteen columns proceed in the order of the partial derivatives with regard to

$$\xi_i,\ \xi_j,\ \xi_k,\ \eta_{ij},\ \eta_{jk},\ \eta_{ki},\ \lambda_{ij},\ \lambda_{jk},\ \lambda_{ki},\ \beta_{ij},\ \beta_{jk},\ \beta_{ki},\ \beta_{ji},\ \beta_{kj},\ \beta_{ik}, \tag{67}$$

respectively, and the rows in the order of the respective equations

$$A_{ijk},\ A_{jki},\ A_{kij},\ B_{ijk},\ B_{jki},\ B_{kij},\ C_{ijk},\ C_{jki},\ C_{kij},\ D_{ijk},\ D_{jki},\ D_{kij},\ E_{ijk},\ E_{jki},\ E_{kij}. \tag{68}$$

Consider now the subdeterminant of the thirteenth order of D formed by cutting out the fifth and sixth columns and dropping the fourteenth and fifteenth rows; designate this subdeterminant by E, and its columns by C_i and rows by R_j, and by $E_{i,j}$ the element common to C_i and R_j.

It is not difficult to isolate a unique non-vanishing term in E and thus prove that E itself does not vanish.

To this end transform the determinant E by the following operations: 1) Replace ξ_j by zero; 2) replace all the β's by zeros except β_{ji}; 3) from C_{10} subtract C_{12}; 4) from R_{12} subtract R_8; 5) from C_{11} subtract $\beta_{ji} C_5$; 6) from C_4 take $\frac{\lambda_{jk}}{\rho} C_1$; 7) from R_{13} take $\frac{\rho}{\lambda_{jk}} R_1$. In the resulting determinant ξ_k occurs only at $E_{3,8}$ and $E_{12,3}$; accordingly we have the coefficient of ξ_k^2 by suppressing R_3, R_8, C_3 and C_{12}. In the thus depleted R_1 of E the term $\lambda_{ki}\eta_{ki}$ occurs only at $E_{10,1}$; in the depleted C_1, $\lambda_{ki}\eta_{ki}$ occurs only at $E_{1,12}$; in the depleted R_2 the term $\lambda_{ij}\eta_{ij}$ occurs only at $E_{8,2}$; and in the depleted C_2 the term $\lambda_{ij}\eta_{ij}$ appears only at $E_{2,10}$; accordingly, by suppressing further from E, R_1, R_2, R_{10}, R_{12}, C_1, C_2, C_8, C_{10}, we have the coefficient of $(\xi_k\lambda_{ki}\eta_{ki}\lambda_{ij}\eta_{ij})^2$; in this last-named coefficient there occurs a unique element, $\beta_{ji}\xi_i$, at $E_{4,13}$ of the original determinant E. By these succes-

sive steps the coefficient of $\beta_{ji}\xi_i(\xi_k\lambda_{ki}\eta_{ki}\lambda_{ij}\eta_{ij})^2$ is made to appear in the following form:

$$\begin{array}{c} \begin{array}{cccccc} C_5 & C_6 & C_7 & C_9 & C_{11'} & C_{13} \end{array} \\ \left|\begin{array}{cccccc} 0 & 0 & 0 & \frac{\rho}{\lambda_{ki}}-1 & \lambda_{ij}-1 & \frac{\rho}{\lambda_{jk}}-1 \\ 0 & 0 & 0 & \lambda_{jk}-1 & \frac{\rho}{\lambda_{ki}}-1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_{ki}-1 \\ 1-\lambda_{ij} & 1-\frac{\rho}{\lambda_{ki}} & 0 & -\beta_{ji} & 0 & \beta_{ji} \\ 1-\frac{\rho}{\lambda_{jk}} & 0 & 1-\lambda_{ki} & 0 & -\beta_{ji} & 0 \\ 1-\frac{\rho}{\lambda_{ki}} & 1-\lambda_{jk} & 0 & 0 & \beta_{ji} & 0 \end{array}\right| \begin{array}{c} R_4 \\ R_5 \\ R_6 \\ R_7 \\ R_9 \\ R_{11} \end{array} \end{array} \qquad (69)$$

Hence we see that the term

$$\beta_{ji}\xi_i\left\{\xi_k\lambda_{ki}\eta_{ki}\lambda_{ij}\eta_{ij}(1-\lambda_{ki})\left|\begin{array}{cc} \lambda_{ij}-1 & \frac{\rho}{\lambda_{ki}}-1 \\ \frac{\rho}{\lambda_{ki}}-1 & \lambda_{jk}-1 \end{array}\right|\right\}^2 \qquad (70)$$

occurs but once in the determinant D.

We may conclude then that the system of equations possesses no more than two independent solutions.

It is not difficult to write out the extension of the argument of the last paragraph to the case of n arbitrary. To this end consider the determinant of the twenty-eighth order arising in the group associated with the problem of five bodies, and write its columns in the order of the partial derivatives with regard to the variables

$$\left.\begin{array}{l} \xi_i,\ \xi_j,\ \xi_k,\ \xi_l,\ \eta_{ij},\ \eta_{jk},\ \eta_{ki},\ \eta_{jl},\ \eta_{li},\ \eta_{kl},\ \lambda_{ij},\ \lambda_{jk},\ \lambda_{ki},\ \lambda_{jl},\ \lambda_{li},\ \lambda_{kl}, \\ \beta_{ij},\ \beta_{jk},\ \beta_{ki},\ \beta_{jl},\ \beta_{li},\ \beta_{kl},\ \beta_{ji},\ \beta_{kj},\ \beta_{ik},\ \beta_{lj},\ \beta_{il},\ \beta_{lk}, \end{array}\right\} \qquad (71)$$

respectively, and its rows in the order of the equations (41), (42), (43), (44), namely

$$\left.\begin{array}{l} 34i,\ 34j,\ 34k,\ 34l,\ 33ij,\ 33jk,\ 33ki,\ 33jl,\ 33li,\ 33kl,\ 35ij,\ 35jk,\ 35ki, \\ 35jl,\ 35li,\ 35kl,\ 35ji,\ 35kj,\ 35ik,\ 35lj,\ 35il,\ 35lk,\ 36ij,\ 36jk,\ 36ki, \\ 36jl,\ 36li,\ 36kl, \end{array}\right\} \qquad (72)$$

respectively.

Consider now the minor of this determinant formed by cutting out $C\eta_{jk}$, $C\eta_{ki}$, $R36jk$, $R36ki$. Retaining the designations of the rows and columns as in the original determinant let us transform this minor by the following steps: 1) From $C\beta_{ki}$ and $C\beta_{kl}$ take $C\beta_{kj}$; 2) from $R36kl$ take $\beta_{lk}\,R35kl$; 3) from $R35ki$ subtract $R35kj$; 4) from $R35kl$ subtract $R35kj$; 5) from $R35kl$ take $R35kj$. In the first four columns of the depleted determinant ξ_k occurs only at $(R35kj,\ C\xi_k)$; in the first column p_{ki} occurs only at $R35ki$; in the second column p_{ij} occurs only at $R35ij$; in the fourth column p_{il} occurs only at $R35il$. From $C\xi_l$ take $C\eta_{li}$; from $C\xi_j$ take $C\eta_{ij}$. We have the coefficient of $\xi_k\,p_{ij}\,p_{ki}\,p_{il}$ by cutting out the first four columns and the rows $R35kj,\ ki,\ ij,\ il$. In the first row p_{ki} occurs only at $C\beta_{ki}$; in the second row p_{ij} occurs only at $C\beta_{ij}$; in the fourth row p_{il} occurs only at $C\beta_{il}$; in the first four rows ξ_k occurs only at $(R34k,\ C\beta_{kj})$. We have the coefficient of $\xi_k^2\,p_{ij}^2\,p_{ki}^2\,p_{il}^2$ by cutting off further the first four rows, and the columns $C\beta kj,\ ki,\ ij,\ il$. In the last-named coefficient ξ_k occurs only at $(R36kl,\ C\beta_{kl})$ with multiplier λ_{kl}, and at $(R35kl,\ C\eta_{kl})$; at $(R36li,\ C\beta_{jl})$ we have the term p_{ij} multiplied by $\left(-\frac{\rho_{lij}}{\lambda_{jl}}\right)$, and p_{ij} occurs at no other point in that row or column; at $(R35jl,\ C\eta_{li})$ we have the term p_{ij} multiplied by $\left(-\frac{\lambda_{jl}}{\rho_{ijl}}\right)$, and p_{ij} occurs at no other point in that row or column; cut out then additionally $R36kl$, $R35kl$, $R36li$, $R35jl$, $C\beta_{kl}$, $C\eta_{kl}$, $C\beta_{jl}$, $C\eta_{li}$, and we have the coefficient of $\lambda_{kl}\,\xi_k^4\,p_{ij}^4\,p_{ki}^2\,p_{il}^2$; in this coefficient $\xi_i\,\beta_{ji}$ is a unique term at $(R36ij,\ C\eta_{ij})$, and $\xi_j\,\beta_{ij}$ is a unique term and at $(R36jl,\ C\eta_{jl})$; accordingly, we have a unique term $\lambda_{kl}\,\beta_{ji}\,\beta_{ij}\,\xi_i\,\xi_j\,\xi_k^4\,p_{ij}^4\,p_{ki}^2\,p_{il}^2$ whose coefficient is the determinant formed of the elements common to

$$\left.\begin{matrix} R35jk,\ li,\ ji,\ ik,\ lj,\ lk, & 33ij,\ jk,\ ki,\ jl,\ li,\ kl, \\ C\beta jk,\ li,\ ji,\ ik,\ lj,\ lk, & \lambda ij,\ jk,\ ki,\ jl,\ li,\ kl. \end{matrix}\right\} \tag{73}$$

Examining this twelfth order determinant we remark: 1) The elements of the minor

$$\left.\begin{matrix} R33 \\ C\lambda \end{matrix}\right\} ij,\ jk,\ ki,\ jl,\ li,\ kl \tag{74}$$

are all zero; 2) as regards the elements of the minor

$$\left.\begin{matrix} R35 \\ C\beta \end{matrix}\right\} jk,\ li,\ ji,\ ik,\ lj,\ lk, \tag{75}$$

each involves the β's and λ's only; 3) the minors

$$\begin{matrix} R33\ ij, jk, ki, jl, li, kl; \\ C\beta\ jk, li, ji, ik, lj, lk \end{matrix} \tag{76}$$

and

$$\begin{matrix} R35\ jk, li, ji, ik, lj, lk; \\ C\lambda\ ij, jk, ki, jl, li, kl \end{matrix} \tag{77}$$

involve the λ's only, and are equal, the rows of one being the columns of the other, and reciprocally.

Accordingly we seek to determine whether the determinant

$$\begin{vmatrix} 1-\frac{\rho_{ijk}}{\lambda_{ki}} & 0 & 1-\lambda_{ij} & 1-\frac{\rho_{ijk}}{\lambda_{jk}} & 0 & 0 \\ 1-\lambda_{jk} & 0 & 1-\frac{\rho_{ijk}}{\lambda_{ki}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-\lambda_{ki} & 0 & 0 \\ 1-\frac{\rho_{jkl}}{\lambda_{kl}} & 1-\frac{\rho_{jli}}{\lambda_{ij}} & 1-\frac{\rho_{jli}}{\lambda_{li}} & 0 & 1-\lambda_{jl} & 1-\frac{\rho_{jkl}}{\lambda_{jk}} \\ 0 & 1-\lambda_{li} & 0 & 1-\frac{\rho_{lik}}{\lambda_{kl}} & 1-\frac{\rho_{lij}}{\lambda_{ij}} & 1-\frac{\rho_{lik}}{\lambda_{ki}} \\ 0 & 1-\frac{\rho_{kli}}{\lambda_{ki}} & 0 & 0 & 1-\frac{\rho_{jkl}}{\lambda_{jk}} & 1-\lambda_{kl} \end{vmatrix} \tag{78}$$

vanishes or not.

This last determinant may be written as follows:

$$(1-\lambda_{ki})\begin{vmatrix} 1-\frac{\rho_{ijk}}{\lambda_{ki}} & 1-\lambda_{ij} \\ 1-\lambda_{jk} & 1-\frac{\rho_{ijk}}{\lambda_{ki}} \end{vmatrix} \cdot \begin{vmatrix} 1-\frac{\rho_{jli}}{\lambda_{ij}} & 1-\lambda_{jl} & 1-\frac{\rho_{jkl}}{\lambda_{jk}} \\ 1-\lambda_{li} & 1-\frac{\rho_{lij}}{\lambda_{ij}} & 1-\frac{\rho_{lik}}{\lambda_{ki}} \\ 1-\frac{\rho_{kli}}{\lambda_{ki}} & 1-\frac{\rho_{jkl}}{\lambda_{jk}} & 1-\lambda_{kl} \end{vmatrix}, \tag{79}$$

in which form its non-vanishing is obvious.

For the general case it is in similar manner made to appear that there is a

non-vanishing minor of order $2n^2-n-2$ containing the unique term made up of the product of the following terms:

$$(\lambda_{s_1s_4}\lambda_{s_1s_5}\ldots\lambda_{s_1s_n})(\xi_{s_2}\beta_{s_3s_2}\xi_{s_3}\beta_{s_4s_3}\ldots\xi_{s_{n-1}}\beta_{s_ns_{n-1}})(p^2_{s_1s_2}\xi^{2(n-2)}_{s_1}p^{2(n-2)}_{s_2s_3}p^{2(n-3)}_{s_2s_4}\ldots p^2_{s_2s_n}) \quad (80)$$

$$(1-\lambda_{s_1s_2})^2\cdot\begin{vmatrix}1-\lambda_{s_1s_3} & 1-\dfrac{\rho_{s_1s_2s_3}}{\lambda_{s_1s_2}}\\ 1-\dfrac{\rho_{s_1s_2s_3}}{\lambda_{s_1s_2}} & 1-\lambda_{s_2s_3}\end{vmatrix}\cdot\prod_{i=4}^{i=n}\begin{vmatrix}1-\lambda_{s_1s_i} & 1-\dfrac{\rho_{s_2s_1s_i}}{\lambda_{s_2s_1}} & \ldots\ldots & 1-\dfrac{\rho_{s_{i-1}s_1s_i}}{\lambda_{s_{i-1}s_1}}\\ 1-\dfrac{\rho_{s_1s_2s_i}}{\lambda_{s_1s_2}} & 1-\lambda_{s_2s_i} & \ldots\ldots & 1-\dfrac{\rho_{s_{i-1}s_2s_i}}{\lambda_{s_{i-1}s_2}}\\ 1-\dfrac{\rho_{s_1s_3s_i}}{\lambda_{s_1s_3}} & 1-\dfrac{\rho_{s_2s_3s_i}}{\lambda_{s_2s_3}} & \ldots\ldots & 1-\dfrac{\rho_{s_{i-1}s_3s_i}}{\lambda_{s_{i-1}s_3}}\\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots\\ 1-\dfrac{\rho_{s_1s_{i-1}s_i}}{\lambda_{s_1s_{i-1}}} & 1-\dfrac{\rho_{s_2s_{i-1}s_i}}{\lambda_{s_2s_{i-1}}} & \ldots\ldots & 1-\lambda_{s_{i-1}s_i}\end{vmatrix} \quad (81)$$

where p_{ij} is written short for $\lambda_{ij}\eta_{ij}-\beta_{ij}\beta_{ji}$.

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